PLASTIC TORSION AND RELATED PROBLEMS

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Abstract. A simple algorithm to calculate the maximum torsional load for a cylindrical shaft is presented. The algorithm is based on the notion of viscosity solutions to the eikonal equation, and is not restricted to simply-connected cross-sections. Applications to other, related problems, such as ferromagnetic thin films, and elastic buckling of thin film blisters are also discussed.

1. Introduction

A classical problem in Plasticity (more specifically, in Limit Analysis) is that of determining the maximum admissible torsional load for a cylindrical shaft. If the yield criterion is von Mises', and if the cross section is simply connected, the solution can be obtained through Nadai's "sand heap" analogy [8]. When the cross-section is not simply connected, the sand heap construction can still be used, provided that constant heights are judiciously assigned on the contours of the internal holes[1].

Mathematically, constructing sand heaps amounts to solving the eikonal equation. There are many solutions to this equation, typically with discontinuous slopes. However, the tallest admissible sand heap, i.e., the maximal pointwise Lipschitz solution of the eikonal equation can be obtained as its viscosity solution (an introduction to the notion of viscosity solution can be found in [3]). Very efficient computational methods for finding viscosity solutions have become available in the last decade, and they are easy to implement [10].

Inspired by these recent advances, we present an algorithm to compute the limit plastic torque for a cylindrical shaft with multiply connected cross-section. We also discuss the application of the algorithm to a few more physical systems, whose behavior is also modelled by the eikonal equation. We hope that the few lines of Matlab code provided herein will entice many to experiment with plastic torsional loads, isotropic thin film ferromagnets, and elastically buckled thin film blisters.

2. Limit torque for a cylindrical shaft

Let Ω be the cross-section of a cylindrical shaft under torsion, a nice open set in $\mathbb{R}^2$. Since Ω is not assumed to be simply connected, its boundary $\partial\Omega$ may consist of several disjoint components: , $_0$, the outer boundary, and , $_1 = \partial\Omega_1, \dots$, , $_n = \partial\Omega_n$, the boundaries of the internal holes Ω_i. Finally, let $\bar{\Omega}$ denote the "filled" cross-section $\Omega \cup \Omega_1 \cup \dots \Omega_n$.

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[1] See, e.g., [5]. A discussion of the more general elastoplastic problem is in [4].

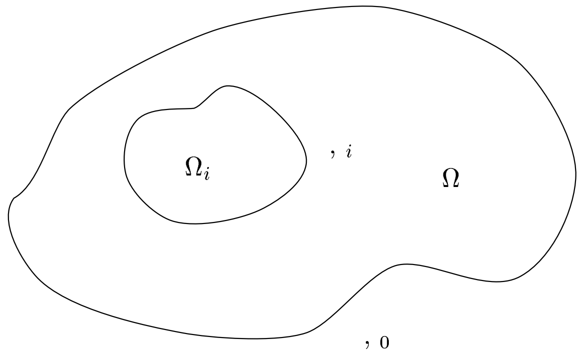

FIGURE 1. The cross-section of a circular shaft.

An equilibrated distribution of shear stresses on Ω is a two-dimensional vector field $\mathbf{t} = (t_1, t_2)$ satisfying

$$\text{div}\,\mathbf{t} = 0 \text{ in } \Omega\,, \quad \mathbf{t}\cdot\nu_i = 0 \text{ on } ,\,_i\,,\ i = 0\ldots, n\,, \tag{2.1}$$

where ν_i is the outer unit normal to $,\,_i$. Such a distribution is plastically admissible (according to von Mises' criterion) if its magnitude never exceeds the yield value. Normalizing this threshold to one, and assuming that the cross-section is everywhere at yield, we are led to seeking divergence-free unit vector fields which are tangent to the boundary. These can be represented using a stream function ψ such that $\nabla\psi = (-t_2, t_1)$. This takes care of the first equilibrium condition in (2.1). The equilibrium conditions at the boundary become $\nabla\psi\cdot\tau_i = 0$, where τ_i is tangent to $,\,_i$, hence ψ is constant on each $,\,_i$. While we are free to assign the value zero on $,\,_0$, the constant values on the inner boundaries are restricted (ψ grows at most with unit slope from the value zero at the outer boundary). Moreover, we want to choose these constants so that the corresponding torque is maximized.

We proceed as follows: Given ψ defined in Ω and constant on each $,\,_i$, let $\bar\psi$ be its extension to $\bar\Omega$ which is constant on each Ω_i. Integration by parts shows that the torque M generated by the shear stress distribution $\mathbf{t}$ is twice the integral over $\bar\Omega$ of the corresponding extended stream function $\bar\psi$:

$$M = \int_\Omega \mathbf{r}(x)\wedge\mathbf{t}(x)\,\mathrm{dx} = -\int_{\bar\Omega}\mathbf{r}(x)\cdot\nabla\bar\psi(x)\,\mathrm{dx} = 2\int_{\bar\Omega}\bar\psi(x)\,\mathrm{dx}\ . \tag{2.2}$$

Here $\mathbf{r}(x) = (x_1, x_2)$ is the radius vector from the origin and $\mathbf{r}\wedge\mathbf{t} = x_1t_2 - x_2t_1$. Thus, the maximal admissible torsional load is obtained by plugging in (2.2) the largest solution $\bar\psi$ to the problem

$$|\nabla\bar\psi(x)| = \chi_\Omega(x) \ \text{ in } \bar\Omega\,, \quad \bar\psi(x) = 0 \text{ on } \partial\Omega\,, \tag{2.3}$$

where $\chi_\Omega(x)$ equals one on Ω and zero elsewhere.

3. LEVEL SET METHODS AND VISCOSITY SOLUTION TO THE EIKONAL EQUATION

Equation (2.3) is the eikonal equation (the name comes from the fact that it describes light propagation: here we are assuming infinite speed of propagation in the holes). Depending on Ω, it may have no smooth solution, but many different ones with discontinuous slopes. The theory of viscosity solutions gives a way to break this degeneracy by singling out the maximal pointwise Lipschitz solution to (2.3)

```
% Solution to the Eikonal Equation, two test-cases on a disk
% F. Alouges & A. DeSimone July 1999
%
close all;clear all;

big=1.e9;
small=1.e-40;
M=100;N=100;
x=[0:N-1]/(N-1);
[x,y]=meshgrid(x,x);
S = (1.-sign((x-0.5).^2+(y-0.5).^2-0.25))/2.; % The disk
T = big*S;

val = menu('Choose a test-case','Torsion','Magnetism');
if (val==1)
   % Torsion. Take infinite speed inside a square hole
   S(M/4:M/2,N/4:N/2) = big;
else
   T(:,M/2) = 0.; % Magnetic football
end

C = 1./(S(2:M-1,2:N-1)+small); % avoids division by 0 outside the domain
C2 = C.^2;

% Main loop
for i=1:max(M,N)
   A = min(T(1:M-2,2:N-1),T(3:M,2:N-1));
   B = min(T(2:M-1,3:N),T(2:M-1,1:N-2));
   Delta = 2*C2-(A-B).^2;
   SGN = (sign(Delta)+1.)/2.;
   X1 = 0.5*(A+B+sqrt(abs(Delta)));
   X2 = min(A,B) + C;
   T(2:M-1,2:N-1) = min(T(2:M-1,2:N-1),SGN.*X1+(1.-SGN).*X2);
end

if (val == 1)
% FIGURE 1 : 3D Graph of the potential
   figure(1); surf(T);
% FIGURE 2 : Contour lines
   figure(2); contour(T,20);
   axis equal; axis([0 M 0 N]);
   Sand_Heap_Volume=abs(sum(sum(T)))
else
% FIGURE : KERR like visualisation
   P=(T(2:M,2:N)-T(1:M-1,1:N-1));
   pmax=max(max(P));pmin=min(min(P));
   P=uint8(S(1:M-1,1:N-1).*(64*(P-pmin)/(pmax-pmin)+1));
   image(P);
   colormap gray;
   map = colormap;
   map(1,:)=[0 1 1]; % First value for the outside
   colormap (map)
   axis square
end
```

FIGURE 2. The MATLAB script.

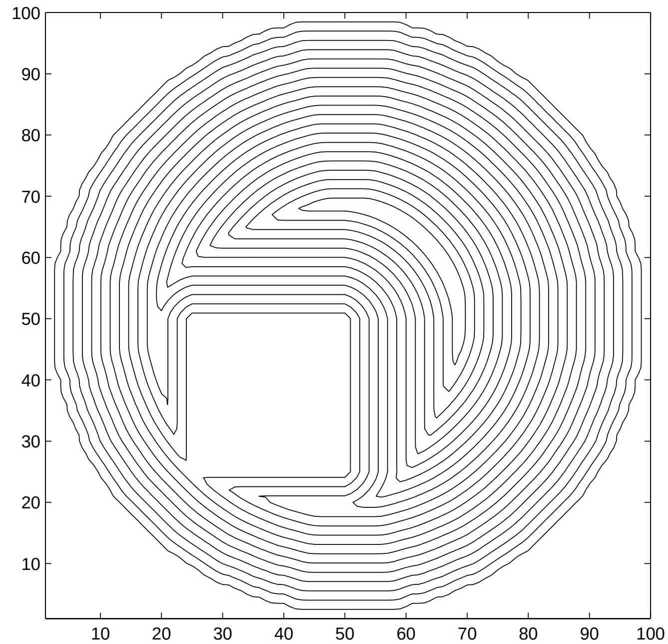

FIGURE 3. Field lines for the shear stresses realizing the maximal admissible torque in a circular cross-section with a square hole.

(see [7], Proposition 7.3)[2]. A very efficient strategy for the numerical computation of viscosity solutions is described in [10]. The basic idea of the algorithm is to view $\bar{\psi}(x)$ as the first arrival time at x of a light ray emanating from the outer boundary, and to construct the solution by propagating a wave front inwards, starting from $\partial\bar{\Omega}$. The algorithm can be made very fast by updating information only in a narrow band around the marching front. We have chosen not to implement this feature in our Matlab script, see Figure 2, to keep the lines of code to a minimum. Figure 3 shows the field lines for $\mathbf{t}$ (the value of M is easily obtained from the integral of $\bar{\psi}$, see (2.2), which is printed out by the program) in a simple yet representative example.

4. Related problems

The plastic torsion problem gives just one example where the eikonal equation

$$|\nabla\psi(x)| = 1 \text{ in } \Omega\,, \quad \psi(x) = 0 \text{ on } \partial\Omega\,, \tag{4.1}$$

models the response of a physical system. We give two more.

In the absence of an applied magnetic field, the magnetization in a thin ferromagnetic film of cross section Ω made of a soft ferromagnetic material can be thought of as a two-dimensional divergence-free unit vector field (after normalization) [6]. Introducing a stream function ψ, we get (4.1). For Ω a disk, the viscosity

[2]In fact, Proposition 7.3 in [7] also shows that the viscosity solution is the maximal Lipschitz function satisfying (2.3) with $\leq$ replacing the equality sign. This proves the intuitive idea that, by allowing *all* plastically admissible stress fields rather than only those for which the cross-section is everywhere at yield, we obtain the same maximal torque.

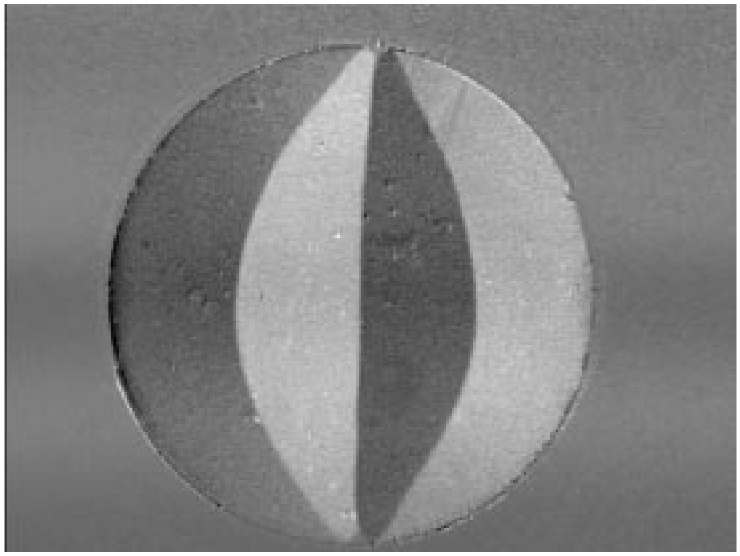

FIGURE 4. Kerr microscopy of a four-domain pattern in a soft ferromagnetic film (courtesy of R. Schäfer, IFW Dresden.)

solution gives a cone which, in terms of field lines for the magnetization vector, represents a vortex. To obtain a pattern similar to the experimentally observed football shaped configuration of Figure 4, we assign to ψ the value zero along a diameter (the output will appear by executing the Matlab program in Figure 2). The reader is encouraged to experiment with our code to find an initial assignement for ψ that will produce the two small triangular regions (closure domains) that are visible in Figure 4.

Another interesting example where (4.1) gives insight in the behavior of a physical system is the formation of buckling-induced blisters in the thin-film coating of a flat substrate. When deposition of the film occurs at a high temperature, a difference in the thermal expansion coefficients between film and substrate may induce, upon cooling, a state of isotropic compression in the film. Delamination of the film and subsequent buckling is then a mechanism to release these compressive stresses. This is actually observed, and it results in the formation of blisters in the coating, with very characteristic shapes. A simplified model has been derived in [9] from von Kármán's theory of plates, leading to (4.1). In this case, ψ is the out-of-plane deflection of the film, $\nabla\psi$ is the slope of the film with respect to the horizontal substrate, and 1 is a (normalized) preferred slope for the film which accomplishes optimal release of the thermal stresses. It is remarkable how such simple a model can faithfully reproduce the complicated folding patterns of the observed blisters. Figure 5 shows the result of our algorithm on a domain whose boundary was scanned in from [1].

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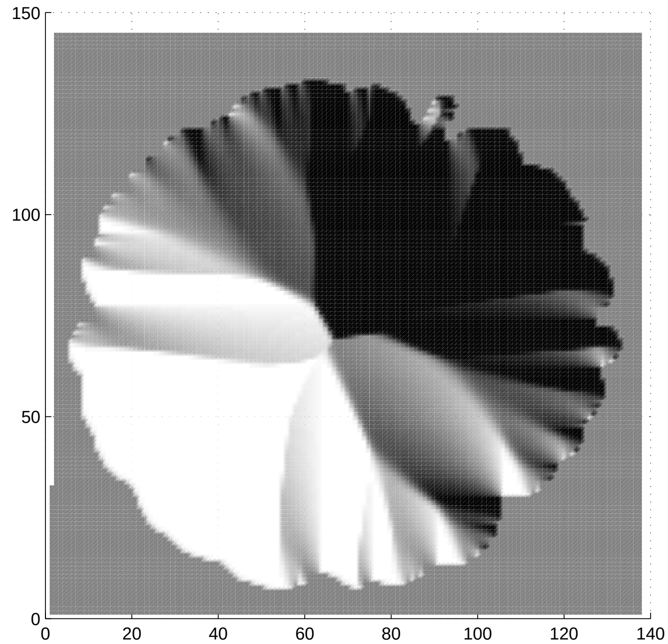
150
100
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